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Characterization of compactly supported refinable splines whose shifts form a Riesz basis

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Abstract

Based on (J. Approx. Theory 86 (1996) 240), we prove that the integer shifts of a multivariate block-wise polynomial $\phi(x)$ which is compactly supported and m-refinable form a Riesz basis if and only if $\phi(x) = cB(x - n - \frac{l}{m-1}|v_1, v_2, \ldots, v_k)$. Here $n, l \in \mathbb{Z}^s$, $c \neq 0$ is a constant, $B(x|v_1, v_2, \ldots, v_k)$ is a multivariate box spline and the matrix $(v_1, v_2, \ldots, v_k)$ is unimodular.

Keywords: Multivariate spline; Refinement; Compact support; Riesz basis; Box spline

1. Introduction

For an integer $m \geqslant 2$, a compactly supported function ϕ is called *m-refinable* if there exists a finite sequence $\{a_n\}$ such that

$$\phi(x) = \sum a_n \phi(mx - n), \quad x \in \mathbb{R}^s. \tag{1}$$

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A function ϕ is called *refinable* if it is m-refinable for some $m \geqslant 2$. A Laurent polynomial $A(z)$ is said to be *m-closed* if $A(z^m)/A(z)$ is a Laurent polynomial. Similarly, $A(z)$ is said to be *polynomial m-closed* if both $A(z)$ and $A(z^m)/A(z)$ are polynomials.

In [6], Lawton et al. found the characterization of a compactly supported refinable univariate spline $\phi(x)$ and proved that $\phi(x)$ is m-refinable if and only if there exists a characterization polynomial $p(z) = \sum_n p_n z^n$ such that $p(z)(z-1)^{k+1}$ is m-closed and $\phi(x) = \sum_n p_n B(x - n - \frac{l}{m-1})$, where $B(x)$ is a B-spline and l is an integer. This result has been extended to multivariate cases by Sun [8] (see Theorem 1). Goodman gave a summary on refinable splines(include refinable vector splines) in [3]. The authors of [6] also pointed out that the $\left\{\phi(x-n)\right\}_{n\in\mathbb{Z}}$ form a Riesz basis if and only if $p(z)$ is a monomial.

For $v_1, v_2, \ldots, v_k \in \mathbb{Z}^s$, one can define a s-dimensional box spline $B(x|v_1, v_2, \ldots, v_k)$ according to [7]. Hereafter, for convenience let $B(x) = B(x|v_1, v_2, \ldots, v_k)$.

A function ϕ is called a *blockwise polynomial* if its support is the union of some simplexes and it is a polynomial on every simplex. For a more precise definition refer to [8]. $P(D)$ is said to be a *homogeneous differential operator* if $P(x), x \in \mathbb{R}^s$, is a homogeneous polynomial. Sun proved the following theorem:

Theorem 1 (*Sun [8]*). *Let $s \geqslant 2$ and ϕ be a compactly supported blockwise polynomial. Then ϕ is m-refinable if and only if*

$$\phi(x) = P(D)\left(\sum_n a_n B\left(x - n - \frac{l}{m-1}\right)\right), \tag{2}$$

where $P(D)$ is a homogeneous differential operator, $\left(\sum_n a_n z^n\right)\prod_{j=1}^{k}(z^{v_j}-1)$ is m-closed, $B(x)$ is a box spline, and l is an integer vector.

Sun did not answer under what condition the integer shifts of the compactly supported refinable spline form a Riesz basis, which is important to the construction of MRA. In this paper, we give this answer (see Theorem 2).

Notation: For $n, l \in \mathbb{Z}^s$, then $n < l$ denotes $n_j < l_j$, $j = 1, 2, \ldots, s$; and $n > l$ and $n = l$ are defined similarly; $x_+ = ((x_1)_+, (x_2)_+, \ldots, (x_s)_+)$. Let $T^s = \{z = (z_1, z_2, \ldots, z_s) \in \mathbb{C}^s \,||z_1| = |z_2| = \cdots = |z_s| = 1\}$.

2. Main result

Let $k \geqslant s$, $v_1, v_2, \ldots, v_k \in \mathbb{Z}^s$. We say the matrix $v = (v_1, v_2, \ldots, v_k)$ is *unimodular* if any matrix formed by any s linearly independent column vectors of the matrix v has determinant value ± 1.

Theorem 2. *Let $s \geqslant 2$ and ϕ be a compactly supported blockwise polynomial. Then ϕ is m-refinable and its integer shifts $\{\phi(x-n)\}_{n\in\mathbb{Z}^s}$ form a Riesz basis if and only if*

$$\phi(x) = cB\left(x - n - \frac{l}{m-1}\right), \tag{3}$$

where $B(x)$ is a box spline, $(v_1, v_2, \ldots, v_k)$ is unimodular, $c \neq 0$ is a constant, and $n, l \in \mathbb{Z}^s$.

2.1. Preliminaries

Lemma 3. *Let $B(x)$ be a box spline and $x_0 \in \mathbb{R}^s$. Then $\{B(x - x_0 - n)\}_{n\in\mathbb{Z}^s}$ form a Riesz basis if and only if the matrix $(v_1, v_2, \ldots, v_k)$ is unimodular.*

This lemma can be obtained from [2,1,4,5, Theorem 5.1].

In this subsection all the 'm-closed' mean 'polynomial m-closed'.

Proposition 4. *If $p(z)$ is m-closed, then $p(z)$ is m^k-closed, where $k \in \mathbb{N}$.*

Lemma 5. *Let $s = 1$ and $p(z)$ be m-closed, then $p(z)$ has no root on the unit circle if and only if $p(z)$ is a non-zero monomial.*

One can prove this lemma based on [6, Definition 2.1, Lemma 2.3].

We say that $p(z)$ is monomial about variable z_1 if it is the product of a monomial in z_1 and a polynomial in $z_2, \ldots, z_s$.

Lemma 6. *Let $s > 1$ and $z \in \mathbb{C}^s$. If $p(z)$ is m-closed and is not monomial about variable z_1, then there exist $k \in \mathbb{N}$, and a constant $\alpha \in \mathbb{C}$, where $|\alpha| = 1$, such that $p(z_1, z_2, \ldots, z_{s-1}, \alpha)$ is m^k-closed and is not monomial about variable z_1.*

Proof. Since $p(z)$ is not monomial about variable z_1, $p(z) = \sum_{l\in\Lambda} p_l(z_s) \prod_{j=1}^{s-1} z_j^{l_j}$ holds, where $\Lambda = \{l = (l_1, l_2, \ldots, l_{s-1}) \in \mathbb{Z}^{s-1} | p_l(z_s) \neq 0\}$, and $\{l_1 | l \in \Lambda, l = (l_1, l_2, \ldots, l_{s-1})\}$ is a finite set with at least two elements. If the polynomial $Q(z_s) = \prod_{l\in\Lambda} p_l(z_s)$ has degree n, then it has at most n different roots. For k with $m^k - 1 > n$, the polynomial $z_s^{m^k-1} - 1$ has $m^k - 1$ different roots, and so has a root α such that $Q(\alpha) \neq 0$, i.e., for all $l \in \Lambda$, $p_l(\alpha) \neq 0$. So $\{l_1 | p_l(\alpha) \neq 0, l \in \Lambda\}$ is a finite set with at least two elements. Then $p(z_1, z_2, \ldots, z_{s-1}, \alpha) = \sum_{l\in\Lambda} p_l(\alpha) \prod_{j=1}^{s-1} z_j^{l_j}$ is not a monomial about variable z_1.

Since $p(z)$ is m-closed, $p(z)$ must be m^k-closed. So

$$\frac{p(z_1^{m^k}, z_2^{m^k}, \ldots, z_{s-1}^{m^k}, \alpha)}{p(z_1, z_2, \ldots, z_{s-1}, \alpha)} = \frac{p(z_1^{m^k}, z_2^{m^k}, \ldots, z_{s-1}^{m^k}, \alpha^{m^k})}{p(z_1, z_2, \ldots, z_{s-1}, \alpha)}$$

is a polynomial, that is, $p(z_1, z_2, \ldots, z_{s-1}, \alpha)$ is m^k-closed. □

Proposition 7. *If $q(z)$ is an m-closed polynomial which is not a monomial, then it has a zero in T^s.*

Proof. Since $q(z)$ is not monomial, there is a variable z_i such that $q(z)$ is not a monomial about variable z_i. Let $p(z) = q(z_i, z_2, \ldots, z_{i-1}, z_1, z_{i+1}, \ldots, z_s)$, then $p(z)$ is m-closed and $p(z)$ is not monomial about variable z_1. Using Lemma 6 repeatedly, there exist k_s, k_{s-1},

$\ldots, k_2 \in \mathbb{N}$, $\beta_s, \beta_{s-1}, \ldots, \beta_2 \in \mathbb{C}$ and $|\beta_s| = |\beta_{s-1}| = \cdots = |\beta_2| = 1$, such that $p(z_1, \beta_2, \beta_3, \ldots, \beta_s)$ is $m^{\prod_{j=2}^{s} k_j}$-closed and is not monomial. And from Lemma 5, there exists β_1, $|\beta_1| = 1$, such that

$$p(\beta_1, \beta_2, \beta_3, \ldots, \beta_s) = 0.$$

For $\alpha = (\beta_i, \beta_2, \ldots, \beta_{i-1}, \beta_1, \beta_{i+1}, \ldots, \beta_s)$, obviously, $q(\alpha) = 0$ holds. □

Proposition 8. *For $t = (t_1, t_2, \ldots, t_s) \in \mathbb{Z}^s$, $p(t) = \prod_{j=1}^{a} t_{n_j} - c$ is prime, where $c \neq 0$ is a constant and $n_j \neq n_i$, when $i \neq j$.*

Proof. If $p(t)$ is not prime, then $p(t) = \prod_{k=1}^{l} p_k(t)$, where $p_k(t),\ k = 1, 2, \ldots, l$, are prime and have the following two properties: (a) the maximal degree of $p_k(t)$ in each variable is 1; (b) for $j \neq i$, $p_j(t)$ and $p_i(t)$ have no common variable. The above properties can be proved by reduction to absurdity. If $p_k(t)$ does not satisfy (a) or (b), then the degree of $\prod_{k=1}^{l} p_k(t)$ in some variable must be larger than 1. Consequently there exist $q_1(t)$, $q_2(t)$ and $q_3(t)$ none of which include the variable t_{n_1}, such that $\prod_{j=1}^{a} t_{n_j} - c = q_2(t)q_1(t)t_{n_1} - q_2(t)q_3(t)$. Putting $t_{n_1} = 0$ gives $q_2(t)q_3(t) = c$. Clearly, $q_2(t)$ and $q_3(t)$ are constant. So $\prod_{j=1}^{a} t_{n_j} - c$ cannot be written the production of the two nontrivial polynomials. This contradicts the assumption. □

Proposition 9. *If $l \neq 0 \in \mathbb{Z}^s$ and $p(z) = (z^{(l)_+} - z^{(-l)_+})$, then $p(z^m)/p(z)$ is a polynomial and its every prime factor has a root on T^s.*

Proof. We can see that $p(z^m)/p(z)$ is a polynomial by verifying directly

$$p(z^m)/p(z) = \sum_{j=0}^{m-1} ((z^{(-l)_+})^{m-1-j} (z^{(l)_+})^j) = \prod_{j=1}^{m-1} (z^{(l)_+} - \beta_j z^{(-l)_+}),$$

where $\beta_j = \exp(-i2\pi j/m)$, $j = 1, 2, \ldots, m-1$. Let us prove the last assertion. There are three cases: $(-l)_+ = 0$; $(l)_+ = 0$; $(l)_+ \neq 0$ and $(-l)_+ \neq 0$. We only prove the first case, as the others are similar. Now,

$$p(z^m)/p(z) = \prod_{j=1}^{m-1} (z^{(l)_+} - \beta_j).$$

We only need to prove that any prime factor of $r(z) = z^{(l)_+} - \beta_j$ has a root on T^s for $j = 1, 2, \ldots, m-1$. Obviously $l_+ \neq 0$ sine $l \neq 0$ and $(-l)_+ = 0$. Assume the number of elements in the set $\{l_j \neq 0, 1 \leqslant j \leqslant s\}$ is a and $l_{n_1}, l_{n_2}, \ldots, l_{n_a}$ denote these elements. Then $z^{(l)_+} = z_{n_1}^{l_{n_1}} z_{n_2}^{l_{n_2}} \cdots z_{n_a}^{l_{n_a}}$. Let α be the least common multiple of $l_{n_1}, l_{n_2}, \ldots, l_{n_a}$, $z_{n_1} = \beta_j^{1/l_{n_1}} t_{n_1}^{\alpha/l_{n_1}}$, $z_{n_2} = t_{n_2}^{\alpha/l_{n_2}}$, $z_{n_3} = t_{n_3}^{\alpha/l_{n_3}}, \ldots, z_{n_a} = t_{n_a}^{\alpha/l_{n_a}}$, and let $z_j = t_j$ when $j \neq n_k$, $k = 1, 2, \ldots, a$. Then

$$r(z(t)) = (z^{(l)_+} - \beta_j) = \beta_j((t_{n_1} \cdots t_{n_a})^{\alpha} - 1) = \beta_j \prod_{k=0}^{\alpha-1} \left(\prod_{p=1}^{a} t_{n_p} - \gamma_k \right),$$

where $\gamma_k = \exp(-i2\pi k/\alpha)$, $k = 0, 1, \ldots, \alpha - 1$. Putting $t_{n_1} = \gamma_k$, $t_{n_2} = t_{n_3} = \cdots = t_{n_a} = 1$, shows that $\prod_{p=1}^{a} t_{n_p} - \gamma_k$ has root on T^s for $k = 0, 1, \ldots, \alpha - 1$. From Proposition 8 we know $\prod_{p=1}^{a} t_{n_p} - c$ is prime when $c \neq 0$ is a constant. So any prime factor of $r(z(t))$ in variable t has a root on T^s. Clearly any prime factor of $r(z)$ must be a product of some prime factors of $r(z(t))$. Hence, due to the equivalence of $z \in T^s$ and $t \in T^s$, any prime factor of $r(z)$ has a root on T^s. $\square$

Lemma 10. *Let $p(z)$ be a non-zero polynomial, $v_j \neq 0 \in \mathbb{Z}^s$, $j = 1, 2, \ldots, k$ and $p(z)\prod_{j=1}^{k}(z^{(v_j)_+} - z^{(-v_j)_+})$ be m-closed, then $p(z)$ has no root on T^s if and only if $p(z)$ is a monomial.*

Proof. The sufficiency is apparent and we only need to prove the necessity. Let $q_j(z) = (z^{(v_j)_+} - z^{(-v_j)_+})$. From Proposition 9, $\prod_{j=1}^{k} \frac{q_j(z^m)}{q_j(z)}$ is a polynomial and any of its prime factors has a root on T^s. So $p(z)$ and $\prod_{j=1}^{k} \frac{q_j(z^m)}{q_j(z)}$ have no common factor. On the other hand, from the condition that $p(z)\prod_{j=1}^{k} q_j(z)$ is m-closed, we know there is a polynomial $r(z)$ such that

$$r(z) = \frac{p(z^m)\prod_{j=1}^{k} q_j(z^m)}{p(z)\prod_{j=1}^{k} q_j(z)} = \frac{p(z^m)\prod_{j=1}^{k} \frac{q_j(z^m)}{q_j(z)}}{p(z)}.$$

Therefore, $p(z^m)/p(z)$ is a polynomial, that is, $p(z)$ is m-closed. From Proposition 7, $p(z)$ must be a monomial. $\square$

2.2. *Proof of Theorem 2*

The sufficiency part of the theorem is apparent from Lemma 3, so we only need to prove the necessity.

Based on the fact that $\sum_n B(x-n)$ is a constant and [5, Theorem 5.1] (or [9, Theorem 1.1]), we know the shifts of ϕ cannot form a Riesz basis when the order of the homogeneous differential operator $P(D)$ is not zero.

When the order of the homogeneous differential operator $P(D)$ is zero, from Theorem 1, there exist $M, N \in \mathbb{Z}^s$ such that $\phi(x) = \sum_{N \leqslant n \leqslant M} a_n B(x - n - \frac{l}{m-1}), l \in \mathbb{Z}^s$. Let $A(z) = \sum_{N \leqslant n \leqslant M} a_n z^n$ and $C(z) = \sum_{N \leqslant n \leqslant M} a_n z^{n+(-N)_+}$, then $A(z) = z^{-(-N)_+} C(z)$. Therefore $C(z)\prod_{j=1}^{k}(z^{(v_j)_+} - z^{(-v_j)_+})$ is a polynomial which is m-closed because $A(z)\prod_{j=1}^{k}(z^{v_j} - 1)$ is m-closed.

Now, let us prove that $(v_1, v_2, \ldots, v_k)$ is unimodular. If it is not unimodular, from Lemma 3, $\{B(x-n)\}_{n\in\mathbb{Z}^s}$ is not a Riesz basis. Thus $\sum_{n\in\mathbb{Z}^s} \left|\hat{B}(\omega + 2n\pi)\right|^2$ has a root. Since $\sum_{n\in\mathbb{Z}^s} \left|\hat{B}(\omega + 2n\pi)\right|^2$ is bounded and continuous, and $|C(\exp(-i\omega))|$ is bounded too, we obtain that

$$\begin{aligned}\sum_{n\in\mathbb{Z}^s} \left|\hat{\phi}(\omega+2n\pi)\right|^2 &= |A(z)|^2 \sum_{n\in\mathbb{Z}^s} \left|\hat{B}(\omega+2n\pi)\right|^2 \\ &= |C(z)|^2 \sum_{n\in\mathbb{Z}^s} \left|\hat{B}(\omega+2n\pi)\right|^2\end{aligned}$$

has a root. Thus $\left\{\phi(x-n)\right\}_{n\in\mathbb{Z}^s}$ is not a Riesz basis, which contradicts the condition that $\left\{\phi(x-n)\right\}_{n\in\mathbb{Z}^s}$ forms a Riesz basis. So $(v_1, v_2, \ldots, v_k)$ is unimodular. Furthermore, $\{B(x-n)\}_{n\in\mathbb{Z}^s}$ is a Riesz basis from Lemma 3.

From the above discussion, the sufficient and necessary condition that $\left\{\phi(x-n)\right\}_{n\in\mathbb{Z}^s}$ forms a Riesz basis is that $C(z)$ has no root on T^s. Because $C(z)$ is polynomial m-closed, from Lemma 10, $C(z)$ is a non-zero monomial, i.e., $A(z)$ is a monomial. So $\phi(x) = cB(x-n-\frac{l}{m-1})$, where $c \neq 0$ is a constant. $\square$

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